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Note

A New, but Flawed, Numerical Method for Vortex Patch Evolution in Two Dimensions

1. Discussion

In this note we present reasons for the erroneous results obtained after a short time integration by Buttke's [1] new numerical algorithm. He attempts to compute vortex patch evolution for two-dimensional, inviscid, incompressible flows and claims that tangent–slope discontinuities or curvature singularities arise after a short, finite time on the boundaries of vortex patches [2]. By contrast, contour-dynamics algorithms with curvature-controlled adaptive node adjustment do not exhibit these singularities in longer time integrations using identical initial conditions [3]. Reference [3] argues that such singularities are implausible on basic fluid dynamical grounds.

A decade ago, Zabusky, Hughes, and Roberts [4] introduced contour dynamics, a numerical algorithm for computing the evolution of general piecewise-constant vorticity distributions. In this algorithm, each "boundary" or contour of discontinuous vorticity is represented (discretized) by a finite number of nodes, and the velocity at each node is calculated from discrete sums approximating contour integrals. The set of all contours comprising the piecewise-constant vorticity distribution evolves in direct response to this self-induced velocity field. The algorithm has seen increasingly widespread use as a means of exploring ultra-high Reynolds number flows. For a recent review, see Ref. [5].

Buttke's [1] numerical algorithm differs essentially from contour dynamics by the method of representing the contours from which the velocity field is calculated. This algorithm actually makes use of two bounding contours about each (simply-connected) region of uniform vorticity. Referring to Fig. 1, the first contour (the dashed line) is represented by a finite number of nodes just as in contour dynamics (this will be termed the "nodal boundary"), while the second contour (the solid line) is determined from the first by filling the interior of the nodal boundary with a number of contiguous square computational elements, of dimensions $2^m L$ by $2^m L$, $m = 0, 1, 2, \ldots$, chosen such that the centers of the smallest elements lie next to, but just within, the nodal boundary. The "staircase" edge of this block of elements is termed the "block boundary." The purpose of this is to enable the velocity field to be calculated from a sum over the contributions from each square element, rather than from a mathematically equivalent contour integration around the block boundaries.

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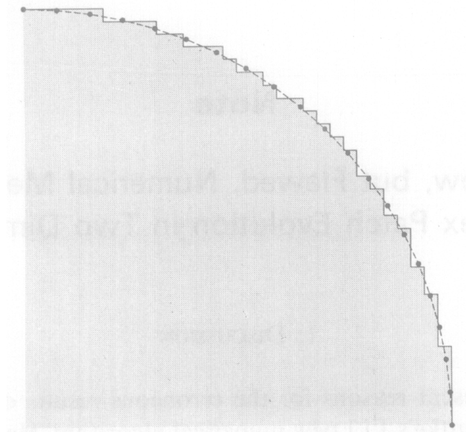

FIG. 1. A sketch showing the two boundaries used in Buttke's computational algorithm around a quarter of a circular patch (as in Fig. 2b of Ref. [1]). The dashed line connecting the nodes gives the nodal boundary, and the solid line gives the block boundary.

It is the peculiar nature of this velocity field which renders the algorithm unsuitable for numerical computation. There are logarithmically-divergent strain rates at all points along each block boundary where it takes a 90° turn [3]. For instance, near the 90° turn shown in Fig. 2,

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \approx -\frac{\omega}{2\pi} \log \frac{r}{L}, \tag{1}$$

where ω is the jump in vorticity upon crossing into the shaded region, r is the dis-

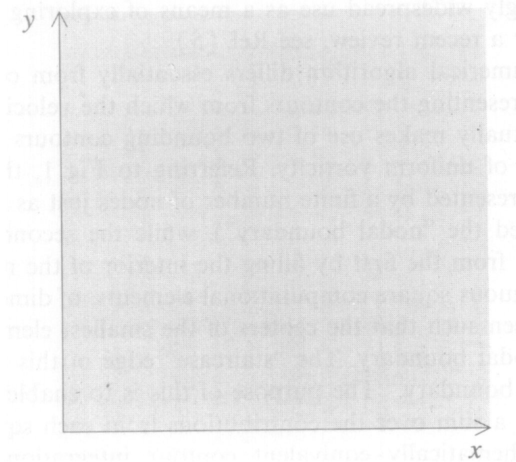

FIG. 2. A local view of a 90° turn.

tance from the 90° turn, and L is the side-length of the smallest square element [3] ($r \ll L$). Hence, if some nodes come close to the 90° turns—this is not an unlikely possibility—the irregular strain field will disturb the trajectories of these nodes in an undesirable manner. The greatest possibility for a node to come close to a 90° turn occurs in the vicinity of a local curvature maximum, for here adjacent nodes may get as close as one quarter of the minimum element side-length [1]. Here indeed Buttke [2] does observe an apparent tangent-slope discontinuity. We are left to conclude that the tangent-slope discontinuities along the block boundaries in effect induce apparent tangent-slope discontinuities along the nodal boundaries. Mesh refinement will not overcome the formation of these erroneous discontinuities, for, from purely geometrical considerations, a reduction in the side-length L of the smallest square element does not alter the probability of a node feeling strain of a given value, since strain is a scale-invariant quantity (in (1) above, note that r and L decrease proportionally).

A final remark is made concerning "style" in computational fluid dynamics. New algorithms employing special formulations (e.g., piecewise-constant vorticity) gain acceptance when they achieve high accuracy with minimal computational cost. Contour dynamics and other Lagrangian curve-tracking algorithms achieve this by using the contour curvature κ to dynamically adjust resolution or redistribute nodes [5]. Curvature, specifically its variation with arclength s, is in fact a direct gauge of the accuracy of a calculation [6]. Buttke [1], however, does not use the curvature to dynamically adjust resolution but instead allows nodes to move freely in response to the computed velocity field. Nor does he show curvature in his computation of a finite-time tangent-slope discontinuity [2], but one can infer from his Fig. 2, a plot of the tangent angle Θ versus arclength s, that the curvature ($d\Theta/ds$) shows signs of containing spurious internodal-scale oscillations. Internodal-scale oscillations indicate the presence of undesirable feedback paths in the discrete system which are not part of the continuum system. It is imperative, therefore, that new algorithms first demonstrate control over the discretization process before attempting to compute delicate fluid motions.

Acknowledgment

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